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A flexible parameterization of the CKM matrix via the singular-value-decomposition method

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Abstract. We investigate a flexible method by which we can test the unitarity of the quark flavor-mixing matrix step by step. Singular-value-decomposition (SVD) techniques are used in analyzing the mixing matrix over a broader parameter region than the unitarity region. Unitary constraints let us extract CP-violating properties without any specific parameterization when the magnitudes of at least three mixing-matrix elements in three-generation quark mixing are given. This method can also be applied to the analysis of lepton flavor mixing, in which only a few moduli are presently measured.

1 Introduction

The Cabibbo–Kobayashi–Maskawa (CKM) [1,2] matrix makes it possible for us to explain all flavor changing weak decay processes and CP-violating phenomena up to now. Unitarity of the CKM matrix in the standard model (SM) is a unique property that we cannot loosen. We can use any parameterization of the CKM matrix as long as its unitarity is conserved. The original parameterization for three-generation quark mixing is the Kobayashi–Maskawa (KM) parameterization. The standard parameterization proposed by Chau and Keung [3,4] is the product of three complex rotation matrices which are characterized by the three Euler angles, $\theta_{12}, \theta_{13}, \theta_{23}$ and a CP-violating phase δ_{13}. A more widely used one is the Wolfenstein parameterization [5], which was suggested as a simple expansion of the CKM matrix in terms of the four parameters λ, A, ρ and η. It has also been known that the CKM matrix for the three-generation case can be parameterized in terms of the moduli of four of its elements [6]. This four-value-KM (4VKM) parameterization is rephasing invariant and directly related to the measured quantities. In the three-generation case we always need four independent parameters to define a unitary 3×3 matrix, as explained, e.g. $\theta_{12}, \theta_{13}, \theta_{23}$ and δ_{13}, or λ, A, ρ and η or even only the moduli of any four independent elements of the matrix.

The 4VKM parameterization has several advantages over the other parameterization. This parameterization does not need any specific representations for the mixing angles as long as the CKM is unitary, and no ambiguity over the definition of its complex CP phase is present moreover. Secondly, the Jarlskog invariant quantity J_{CP} and non-trivial higher invariants can be reformulated as functions of moduli and quadri-products [7]. However, in the 4VKM parameterization, the initial four-moduli input values should be fixed by experiments. Once we set four moduli to specific values, the remaining five mixing element moduli are automatically fixed and we may lose some characteristic effects from the interplay between the moduli. From a conceptual point of view it is better if we can reduce the number of a priori experimental input values. This paper presents a novel parameterization in which we start with three moduli input values. Through simple algebraic relations we can determine the remaining six moduli of the mixing elements. With a broader parameter space we can step by step check the compatibility between the measured values of mixing elements and their unitarity properties.

Many groups have made global fits and numerical works on CKM matrix elements with conventional representations which satisfy unitarity [8]. One of the problems in these conventional parameterizations is that they are *fully* and *completely* unitary and are not flexible so as to include possible non-unitary properties resulting from unknown new physics. Therefore, it is a complex task to make a step by step test to check the unitarity with experimental data if you use a unitary parameterization. In the following, we present three extended definitions for the unitarities of the mixing matrix V in the order of the strength of the constraints.

Weak unitary conditions (WUC)

We define the mixing matrix V to be weak unitary if it satisfies

$$\sum_{\alpha} |V_{i\alpha}|^2 = \sum_{j} |V_{j\beta}|^2 = 1 \text{for all} i = u, c, t, \text{and} \beta = d, s, b. \tag{1}$$

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These constraints appear to be well satisfied experimentally for the three-generation case, and we start from this. Actually it was pointed out that there is an apparent functional violation in the available data: $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 < 1$ [9]. For such a case, with $\sum_\alpha |V_{u\alpha}|^2 = a < 1$, we can easily generalize our method, and we just start with this new condition.

Almost unitary conditions (AUC)

In addition to the constraint equation (1), if the following constraints are satisfied:

$$\sum_{\alpha,i\neq j} V_{i\alpha}^* V_{j\alpha} = \sum_{j,\alpha\neq\beta} V_{j\alpha}^* V_{j\beta} = 0 \text{ for some parts of } i,j = u,c,t, \text{ and } \alpha,\beta = d,s,b, \tag{2}$$

let us call the mixing matrix almost unitary. Some combinations, which do not satisfy (2), may not make closed triangles, and may have different areas even though making closed triangles. We have no specific models in which the mixing matrix satisfies these almost unitary conditions. Therefore, we will not consider the case with AUC.

Full unitary conditions (FUC)

This corresponds to the usual unitarity in which (1) and (2) are satisfied *for all the indices*. All six unitarity triangles from (2) have the same areas.

In Sect. 2, we propose an alternative and a more flexible parameterization of the CKM matrix in terms of the three moduli and the one independent parameter which is induced by the singular-value-decomposition (SVD) method. We describe how to get the new parameterization of the CKM matrix by using the SVD method in the three-generation case. Unlike the previous parameterization with four moduli [6], we have a more flexible leverage to test the unitarity step by step. We start with only three moduli rather than four moduli, and the remaining one can be adjusted depending on the condition of the unitarity, which we apply, i.e. WUC or FUC. In Sect. 3, we analyze the CKM matrix numerically with our parameterization with the SVD method. Conclusions are also in Sect. 3. Appendices A and B include details on the SVD method.

2 New parameterization of the CKM matrix by the SVD method

We start with a definition in such a way that it satisfies the weak unitary conditions, (1): we have six constraint equations for the three-generation mixing. These constraints are only parts of the unitarity conditions and the introduced mixing matrix V may not be fully unitary. We study this explicitly with the three-generation quark flavor-mixing matrix V considering their absolute values and choose three independent moduli as starting input parameters. The explicit analysis depends on the choice of the three input parameters. We consider the case with

Our choice (Set A): input parameters $|V_{us}|, |V_{ub}|, |V_{cb}|$.

We can also choose different sets of input parameters, for example:

Set B: input parameters $|V_{ud}|, |V_{us}|, |V_{cd}|$;
Set C: input parameters $|V_{us}|, |V_{cs}|, |V_{cb}|$.

Mathematically the three parameterizations of Set A, B and C are all equivalent if the three input values of each set are independent of each other and all are equally precisely measured. However, in reality, the upper-left 2×2 part of CKM matrix is *approximately* unitary and only one independent variable is dominantly evident, for example, the parameter λ in the Wolfenstein parameterization or the Cabibbo angle in $\sin\theta_c$. Therefore, Set B would be the worst choice for numerical analyses. For our choice of Set A, the three inputs are all off-diagonal and independent of each other, and all three values can be determined by the three semileptonic decays, in which new physics contributions are severely suppressed. Therefore, we select the upper off-diagonal elements in V, namely, $|V_{us}|, |V_{ub}|, |V_{cb}|$ as the initial input variables in our analysis, i.e. the case with Set A.

If we are given the three input values of Set A, then we get the following values $|V_{ud}|$ and $|V_{tb}|$:

$$|V_{ud}|^2 = 1 - |V_{us}|^2 - |V_{ub}|^2, \tag{3}$$
$$|V_{tb}|^2 = 1 - |V_{ub}|^2 - |V_{cb}|^2. \tag{4}$$

To obtain the four remaining elements, $|V_{cd}|, |V_{cs}|, |V_{td}|$ and $|V_{ts}|$, we write the four constraints for these four elements in (1) in matrix form:

$$RX = B, \tag{5}$$

where

$$R = \begin{pmatrix} 1\,1\,0\,0 \\ 0\,0\,1\,1 \\ 1\,0\,1\,0 \\ 0\,1\,0\,1 \end{pmatrix}, \tag{6}$$

$$X = (|V_{cd}|^2, |V_{cs}|^2, |V_{td}|^2, |V_{ts}|^2)^{\mathrm{T}}, \tag{7}$$

$$B = (1 - |V_{cb}|^2, 1 - |V_{tb}|^2, 1 - |V_{ud}|^2, 1 - |V_{us}|^2)^{\mathrm{T}}. \tag{8}$$

In (7) and (8), X and B are column vectors, and T means transpose of the matrix. Because of $\det R = 0$, there is not a unique solution if any. In such a situation there exists a very powerful set of techniques, known as the singular-value-decomposition (SVD) method. The details of this method are given in Appendix A. The remaining mixing elements are expressed as follows:

$$\begin{aligned} |V_{cd}|^2 &= -a + u_1, \\ |V_{cs}|^2 &= a + u_2, \\ |V_{td}|^2 &= a + u_3, \\ |V_{ts}|^2 &= -a + u_4, \end{aligned} \tag{9}$$

where

$$u_1 = \frac{1}{4}(1 + 2|V_{us}|^2 + |V_{ub}|^2 - 2|V_{cb}|^2),$$
$$u_2 = \frac{1}{4}(3 - 2|V_{us}|^2 - |V_{ub}|^2 - 2|V_{cb}|^2), \quad (10)$$
$$u_3 = \frac{1}{4}(-1 + 2|V_{us}|^2 + 3|V_{ub}|^2 + 2|V_{cb}|^2),$$
$$u_4 = \frac{1}{4}(1 - 2|V_{us}|^2 + |V_{ub}|^2 + 2|V_{cb}|^2),$$

and a new variable, "a", is introduced as a coefficient attached to the general solution. If there is no flavor mixing, we can set $a = 1/4$. The value of "a" can be determined from (9) if we know any value of $|V_{cd}|, |V_{cs}|, |V_{td}|, |V_{ts}|$. The constraints of non-negative $|V_{ij}|^2$ are applied for the range of the variable a:

$$a_{\min} = \max(-u_2, -u_3), \quad a_{\max} = \min(u_1, u_4).$$

We note that when the three input values $|V_{us}|, |V_{ub}|, |V_{cb}|$ are given, the moduli squared of the remaining four mixing elements $|V_{cd}|, |V_{cs}|, |V_{td}|, |V_{ts}|$ are just quadratic functions of the parameter a. As "a" increases, $|V_{cs}|$ and $|V_{td}|$ increase, while $|V_{cd}|$ and $|V_{ts}|$ decrease. $|V_{ud}|$ and $|V_{tb}|$ are fixed by the three input values and are independent of the parameter a. The bounds on the parameter "a" will determine the regions of FUC and WUC, which will be explained later.

As a next step, we further assume that the mixing matrix V satisfies the full unitary conditions. Then we have six more constraints:

$$\sum_{j=d,s,b} V_{ij}V_{kj}^* = 0, \quad (i,k) = (u,c),(u,t),(c,t),$$
$$\sum_{j=u,c,t} V_{ji}V_{jk}^* = 0, \quad (i,k) = (d,s),(d,b),(s,b). \quad (11)$$

These constraints cannot be represented analytically without the introduction of complex numbers. If we know all the absolute values of V, however, we can express the necessary and sufficient conditions for the constraints, (11), in a geometric way. Equations (11) give six unitarity triangles corresponding to each six constraints, and all six triangles have an equal area that is directly related to the Jarlskog's rephasing invariant parameter J_{CP}. If we take one of the constraints (11), for example,

$$\sum_{j=u,c,t} V_{jd}V_{jb}^* = 0,$$

a triangle is composed of three sides with lengths $|V_{ud}||V_{ub}|, |V_{cd}||V_{cb}|$, and $|V_{td}||V_{tb}|$, with the necessary condition

$$|V_{cd}||V_{cb}| \le |V_{ud}||V_{ub}| + |V_{td}||V_{tb}|, \quad (12)$$

where the equality holds in the CP-conserving case. For a more general argument, let us rewrite (12) as follows:

$$l_2 \le l_1 + l_3, \quad (13)$$

where, for example, $l_1 = |V_{ud}||V_{ub}|, l_2 = |V_{cd}||V_{cb}|$, and $l_3 = |V_{td}||V_{tb}|$. After taking the square on both sides of (13) we can rearrange the constraint equation as follows:

$$f(l_1,l_2,l_3) \equiv 2l_1^2l_2^2 + 2l_2^2l_3^2 + 2l_1^2l_3^2 - l_1^4 - l_2^4 - l_3^4 \ge 0, \quad (14)$$

where we denote the newly introduced function f for later use. Using Heron's formula, the square of the triangular area can be rewritten as follows:

$$A^2 = s(s-l_1)(s-l_2)(s-l_3) = \frac{1}{16}f(l_1,l_2,l_3), \quad (15)$$

where $s = (l_1+l_2+l_3)/2$. So the necessary condition (14) for the complete triangle means a non-negative value of A^2. Jarlskog's invariant parameter is written as follows:

$$J_{CP} = 2A = \frac{1}{2}\sqrt{f(l_1,l_2,l_3)}. \quad (16)$$

If we expand f in terms of the parameter a, we can write

$$\begin{aligned} f = &-(1-|V_{ub}|^2)^2a^2 \\ &+ 2\Big[|V_{ud}|^2|V_{ub}|^2(|V_{tb}|^2 - |V_{cb}|^2) \\ &\times (|V_{cb}|^2u_1 - |V_{tb}|^2u_3)(|V_{cb}|^2 + |V_{tb}|^2)\Big]a \\ &+ 2|V_{ud}|^2|V_{ub}|^2[|V_{cb}|^2u_1 + |V_{tb}|^2u_3] \\ &- (|V_{cb}|^2u_1 - |V_{tb}|^2u_3)^2 - |V_{ud}|^4|V_{ub}|^4, \end{aligned} \quad (17)$$

where the function f is quadratic in a. We can get the boundaries of the constraint (13), and denote the two roots of the quadratic equation by a_- and a_+ ($> a_-$). Two real roots and the boundary points in the interval depend on only the three input values. The non-existence of real solutions of the quadratic equation means that the three input values do not allow the FUC to be present. We note that if we force the mixing matrix V to be fully unitary, then six triangles from the constraints (11) have the same area, which are the sufficient conditions for the FUC.

We can relate the coefficient a to the CP-violating parameter in another representation of the mixing matrix with the FUC. Let us consider the standard parameterization of the CKM matrix,

$$V_{\rm CKM} = \quad (18)$$
$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix},$$

where $s_{ij} = \sin\theta_{ij}, c_{ij} = \cos\theta_{ij}$. We find that the coefficient a is directly related to the parameters in the standard representation:

$$\begin{aligned} a = &-2s_{12}c_{12}s_{23}c_{23}s_{13}\cos\delta_{13} \\ &- \frac{1}{8}\cos 2\theta_{12}\cos 2\theta_{23}[-3 + \cos 2\theta_{13}]. \end{aligned} \quad (19)$$

The three angles α, β, γ of the unitarity triangle, which characterize CP-violation, are defined as follows:

$$\alpha = \mathrm{Arg}[-(V_{td}V_{tb}^*)/(V_{ud}V_{ub}^*)], \quad (20)$$
$$\beta = \mathrm{Arg}[-(V_{cd}V_{cb}^*)/(V_{td}V_{tb}^*)], \quad (21)$$
$$\gamma = \mathrm{Arg}[-(V_{ud}V_{ub}^*)/(V_{cd}V_{cb}^*)]. \quad (22)$$

Table 1. Input values of the matrix elements and their sources referred from the PDG. The output values are the allowed intervals (95% CL) for WUC and FUC

	Matrix elements	PDG values	Sources	
Input	$\|V_{us}\|$	0.2196 ± 0.0026	K_{e3} decays	
	$\|V_{ub}\|$	$(3.6 \pm 0.7) \times 10^{-3}$	B semileptonic decays	
	$\|V_{cb}\|$	$(41.2 \pm 2.0) \times 10^{-3}$	B semileptonic decays	
	Matrix elements	WUC	FUC	PDG
Output	$\|V_{cd}\|$	$0.210 \sim 0.224$	$0.214 \sim 0.224$	$0.219 \sim 0.226$
	$\|V_{cs}\|$	$0.9735 \sim 0.9768$	$0.9735 \sim 0.9760$	$0.9732 \sim 0.9748$
	$\|V_{td}\|$	$0.004 \sim 0.045$	$0.004 \sim 0.014$	$0.004 \sim 0.014$
	$\|V_{ts}\|$	$0.001 \sim 0.045$	$0.035 \sim 0.045$	$0.037 \sim 0.044$

The sum of those three angles, defined as the intersections of three lines, would be always equal to 180°, even though the three lines may not be closed to make a triangle, i.e. in case that the CKM matrix is not unitary at all. We can also define these quantities from the area of the unitary triangle and its sides:

$$\sin\beta' = \frac{2A}{|V_{td}||V_{tb}||V_{cd}||V_{cb}|}, \tag{23}$$

$$\sin\gamma' = \frac{2A}{|V_{ud}||V_{ub}||V_{cd}||V_{cb}|}, \tag{24}$$

$$\alpha' = \pi - \beta' - \gamma', \tag{25}$$

when the FUC is fully satisfied and the area of the triangles can be defined from (15). Any experimental data that indicate $\alpha \neq \alpha'$ or $\beta \neq \beta'$ or $\gamma \neq \gamma'$ mean that the three-generation quark-mixing matrix V is not fully unitary.

3 Numerical results and discussion

For given input values of $|V_{us}|, |V_{ub}|, |V_{cb}|$, the parameter "$a$" is divided into two regions depending on whether the FUC is satisfied or not. We can divide the range of the parameter into two by setting $l_1 = |V_{ud}||V_{ub}|, l_2 = |V_{cd}||V_{cb}|, l_3 = |V_{td}||V_{tb}|$, for example:

(1) Region I: The maximum among l_1, l_2, l_3 is larger than the sum of the other two values. In other words it is not possible to make any triangle with these three segments. This region is outside of the interval of (a_-, a_+).

(2) Region II: The maximum among l_1, l_2, l_3 is smaller than the sum of the other two values. In other words it is possible to make a unitarity triangle. This region is confined to (a_-, a_+).

In region I, we cannot define J_{CP}. On the contrary we can define J_{CP} in region II and calculate it with l_1, l_2, l_3 as shown in (15). In general region II is surrounded by region I. The two boundary points of region II correspond to the case of the CP-conserving case.

For numerical analyses, we refer to the Particle Data Group (PDG) [4]. Current values of the three input moduli and the corresponding sources of the measured matrix elements are summarized in Table 1. The input values of $|V_{us}|, |V_{ub}|, |V_{cb}|$ are randomly generated within 95% CL with uniform distributions. Each input determines both of the two regions for the WUC and FUC. The WUC is confined to the interval $(a_{\min}, a_{\max})$ of which calculations are described in the previous section. The FUC is confined into the interval (a_-, a_+) which is a subset of $(a_{\min}, a_{\max})$. In the restricted regions we again generated randomly the values of the parameter a for our numeric calculations. Figure 1a shows scattered points for $|V_{td}|$ and $|V_{ts}|$ values when we apply the WUC to the choice of the parameter a. The scattered points compose a quadrant in the $|V_{td}|$–$|V_{ts}|$ plane. In the figure we also draw the curved axis for the parameter a. The labels on the curve are valid only when we set the three inputs to the center values in Table 1. If other input values are taken, the numeric labelling should be slightly changed. Figure 1b presents scattered points when we apply the FUC to the choice of parameter a. The allowed region for FUC is much narrower than that for the WUC and is included in the region for the WUC. The curved axis for the parameter a is identical to that in Fig. 1a. Figure 2 shows the corresponding scattered points for $|V_{ud}|$ and $|V_{cd}|$ when we take the WUC and FUC. The modulus $|V_{ud}|$ does not depend on the parameter a and $|V_{cd}|$ is directly related to the value of a. The axis for the parameter a is, therefore, a vertical line along the axis of $|V_{cd}|$.

In Table 1, we show the numerical output values for the moduli, $|V_{cd}|, |V_{cs}|, |V_{td}|, |V_{ts}|$, within 95% CL with the unitary conditions, WUC and FUC. For comparison, we also show the results of the PDG values. The PDG values of individual matrix elements were determined from the three-level constraints from weak decays of the relevant quarks or from deep inelastic neutrino scattering together with the assumptions of three-generation FUC.

As can be seen, the allowed regions for the WUC are much broader than those for the FUC and the latter are subsets of the former. Our numerical results are consistent with 90% CL on the magnitudes of mixing elements in PDG: $|V_{td}| \simeq 0.004 \sim 0.014$ and $|V_{ts}| \simeq 0.037 \sim 0.044$, particularly. We can see that the FUC and hierarchical input values of $|V_{us}|, |V_{ub}|, |V_{cb}|$ imply $|V_{td}| < |V_{ts}|$. This contrasts with the results of the WUC. In this case, it is possible that $|V_{td}|$ is equal to or even larger than $|V_{ts}|$.

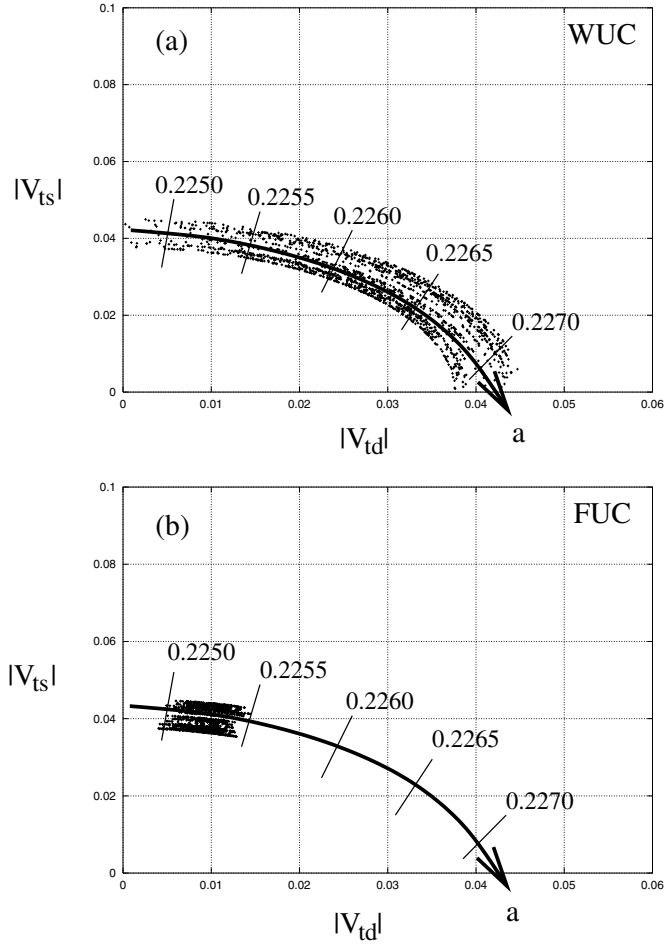

Fig. 1a,b. The scattered plots for $|V_{td}|$ and $|V_{ts}|$ values which satisfy the WUC **a** and FUC **b** in the case of Set A. The allowed points are calculated from the uniformly generated three input values in 95% CL. The curved axis for the parameter "a" is drawn with the centered input values in Table 1, not with the randomly generated input values

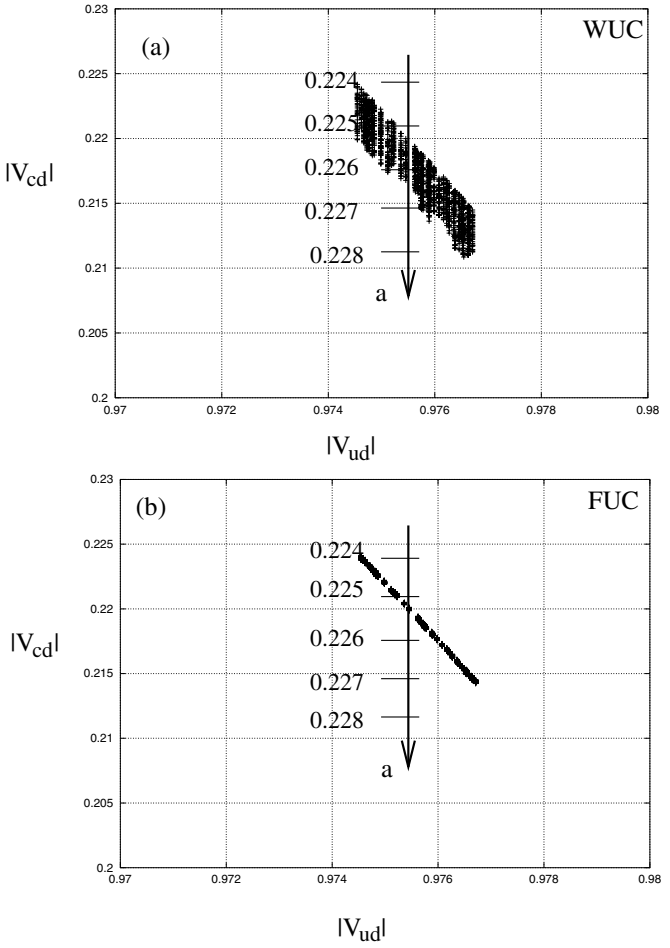

Fig. 2a,b. The scattered plots for $|V_{ud}|$ and $|V_{cd}|$ values which satisfy the WUC **a** and FUC **b** in the case of Set A. The two panels correspond to two panels in Fig. 1

If we start with different mixing elements, like $|V_{ud}|$, $|V_{us}|, |V_{cd}|$ (i.e. Set B), then $|V_{td}|$ and $|V_{ub}|$ are first fixed and we introduce the new parameter b from the SVD method, as shown in Appendix B in detail. The remaining four moduli, $|V_{cs}|, |V_{cb}|, |V_{ts}|$, and $|V_{tb}|$, are dependent on the parameter b and have correlated values. Precise measurements for one of the four moduli will fix the remaining three moduli. Similarly, if we start with $|V_{us}|, |V_{cs}|, |V_{cb}|$ (i.e. Set C), then $|V_{cd}|$ and $|V_{ts}|$ are fixed and the new parameter c will be introduced by following the SVD method. The remaining four moduli, $|V_{ud}|, |V_{ub}|, |V_{td}|$ and $|V_{tb}|$, depend on the parameter c and have correlated values. However, as explained in Sect. 2, it would be much more difficult numerically to perform the analysis with Set B or C compared to Set A due to the approximate unitarity of the upper-left 2×2 part of the CKM matrix.

To conclude, we proposed a flexible method in which the unitarity of the quark-mixing matrix can be tested step by step. The singular-value-decomposition (SVD) method is used in analyzing the mixing matrix over a broader parameter space than the unitary region as well as in presenting a new parameterization of the CKM matrix. The question whether the mixing matrix satisfies the WUC or FUC is a quite difficult and complex matter within the standard PDG parameterization or a similar unitary parameterization. In the parameterization by the SVD method the CKM matrix is represented by three moduli and an additionally induced flexible parameter a [1]. Once the value of the induced parameter a is determined, we can easily distinguish the FUC from the WUC. For example, with the Set A input data, if we get $0.224 \leq a \leq 0.226$, the mixing matrix satisfies the FUC within 95% CL. From Fig. 1b we can also conclude that

[1] In principle, the two methods, 4VKM and SVD, can give identical results. However, in practice the number of input parameters which should be supplied by experiments is reduced from four to three in SVD. This reduction in the number of a priori experimental input values gives conceptually a much better way to analyze the CKM mixing matrix. With the reduced number of input parameters we can check the consistency between the independently measured CKM mixing matrix elements systematically, and we can investigate the inter-relations among the mixing elements by varying a single parameter

the FUC is violated if $|V_{td}| \geq 0.02$ or $|V_{td}| \leq 0.004$. If $|V_{td}| \geq 0.05$, even the WUC is not satisfied. Figure 2b shows that there is a strong correlation between $|V_{ud}|$ and $|V_{cd}|$. This method can also be applied to the analysis of lepton flavor mixing, in which only a few moduli are presently measured.

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Appendix

A The singular-value-decomposition method

For a detailed description of this method we consider the specific case with the input parameters, $|V_{us}|, |V_{ub}|, |V_{cb}|$, which is in Sect. 2. In this case we have to solve (5). According to this method, the matrix R can be decomposed as a product of three matrices:

$$R = UWV^{\mathrm{T}}, \tag{A1}$$

where

$$U = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}, \tag{A2}$$

$$W = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A3}$$

$$V = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{2} & 0 & 1 \\ -1 & 0 & \sqrt{2} & -1 \\ -1 & 0 & -\sqrt{2} & -1 \\ -1 & -\sqrt{2} & 0 & 1 \end{pmatrix}. \tag{A4}$$

In general the matrices U and V are orthogonal in the sense that their columns are orthonormal,

$$\sum_i U_{ik}U_{in} = \delta_{kn}, \tag{A5}$$

$$\sum_i V_{ik}V_{in} = \delta_{kn}. \tag{A6}$$

We note that this decomposition is not unique. For further practical calculations of SVD we refer to [10]. The solutions of (5) are obtained in two types, a special solution and a general solution, which one can get in two different ways. First, the special solution according to SVD is calculated by defining the *inverse* of R as follows:

$$\overline{R} = V[\mathrm{diag}(1/w_{ii})]U^{\mathrm{T}} \tag{A7}$$

$$= \frac{1}{8}\begin{pmatrix} 3 & -1 & 3 & -1 \\ 3 & -1 & -1 & 3 \\ -1 & 3 & 3 & -1 \\ -1 & 3 & -1 & 3 \end{pmatrix}, \tag{A8}$$

where we take $1/w_{ii} = 0$ if $w_{ii} = 0$. The matrix $\overline{R}$ is unique and does not depend on the way how the matrix R is decomposed. The special solution X_{s} is

$$X_{\mathrm{s}} = \overline{R}B. \tag{A9}$$

The inverse matrix $\overline{R}$ does not satisfy the constraints which must be obeyed in the general sense of the inverse, namely, $R\overline{R} \neq \overline{R}R \neq I$. However, it satisfies $R\overline{R}B = B$. Therefore, we can introduce general solutions X_{g} such that

$$RX_{\mathrm{g}} = 0, \tag{A10}$$

and we can add this to the special solution X_{s}. We can see the general solutions of (A10) by simple guessing as follows:

$$X_{\mathrm{g}} = a(-1, 1, 1, -1)^{\mathrm{T}}, \tag{A11}$$

where the coefficient a can take any real value. Actually the coefficient a must be further confined in such a way that the values of the mixing elements V_{ij} should be within the range $[0, 1]$. We can express the complete solutions as

$$X = X_{\mathrm{g}} + X_{\mathrm{s}}. \tag{A12}$$

In algebraic terms, (5) defines R as a linear mapping from the vector space of X to the vector space of B. If R is singular, then there is some subspace of X, called the null space, that is mapped to zero, $RX = 0$. The number of linearly independent vectors that can be found in (A10) is the dimension of the null space called the nullity of R. In the three-generation quark-mixing case the nullity is 1.

B Case with input parameters of Set B

If we take the three independent input parameters $|V_{ud}|$, $|V_{us}|$, and $|V_{cd}|$, we can apply the same procedure to obtain the remaining mixing elements. In this case the matrix (5) becomes $RX = B$ with

$$R = \begin{pmatrix} 1\,0\,1\,0 \\ 0\,1\,0\,1 \\ 1\,1\,0\,0 \\ 0\,0\,1\,1 \end{pmatrix}, \tag{B1}$$

$$X = (|V_{cs}|^2, |V_{cb}|^2, |V_{ts}|^2, |V_{tb}|^2)^{\mathrm{T}}, \tag{B2}$$

$$B = (1 - |V_{us}|^2, 1 - |V_{ub}|^2, 1 - |V_{cd}|^2, 1 - |V_{td}|^2)^{\mathrm{T}}. \tag{B3}$$

Two obvious relations from the weak unitary conditions are

$$|V_{ub}|^2 = 1 - |V_{ud}|^2 - |V_{us}|^2, \tag{B4}$$

$$|V_{td}|^2 = 1 - |V_{ud}|^2 - |V_{cd}|^2. \tag{B5}$$

Following the procedure described in the Appendix A, we can write the total solution of X as follows:

$$|V_{cs}|^2 = b + w_1, \tag{B6}$$
$$|V_{cb}|^2 = -b + w_2, \tag{B7}$$
$$|V_{ts}|^2 = -b + w_3, \tag{B8}$$
$$|V_{tb}|^2 = b + w_4, \tag{B9}$$

where

$$w_1 = \frac{1}{4}(3 - |V_{ud}|^2 - 2|V_{us}|^2 - 2|V_{cd}|^2), \tag{B10}$$
$$w_2 = \frac{1}{4}(1 + |V_{ud}|^2 + 2|V_{us}|^2 - 2|V_{cd}|^2), \tag{B11}$$
$$w_3 = \frac{1}{4}(1 + |V_{ud}|^2 - 2|V_{us}|^2 + 2|V_{cd}|^2), \tag{B12}$$
$$w_4 = \frac{1}{4}(-1 + 3|V_{ud}|^2 + 2|V_{us}|^2 + 2|V_{cd}|^2), \tag{B13}$$

and b is the newly introduced parameter. In this case, if there is no flavor mixing, we can set $b = 1/2$. Equation (B9) shows that any values of $|V_{cs}|, |V_{cb}|, |V_{ts}|, |V_{tb}|$ will determine the value of b. Further constraints are applied for the range of the parameter b by $|V_{ij}|^2 \geq 0$: $b_{\min} = \max(-w_1, -w_4), b_{\max} = \min(w_2, w_3)$. For the FUC we expand f in terms of the parameter b:

$$\begin{aligned} f = & -(1 - |V_{ud}|^2)^2 b^2 \\ & + 2\Big[|V_{ud}|^2 |V_{ub}|^2 (|V_{td}|^2 - |V_{cd}|^2) \\ & \times (|V_{cd}|^2 w_2 - |V_{td}|^2 w_4)(|V_{cd}|^2 + |V_{td}|^2)\Big] b \\ & + 2|V_{ud}|^2 |V_{ub}|^2 [|V_{cd}|^2 w_2 + |V_{td}|^2 w_4] \\ & - (|V_{cd}|^2 w_2 - |V_{td}|^2 w_4)^2 - |V_{ud}|^4 |V_{ub}|^4. \end{aligned} \tag{B14}$$

Like the previous case we can get the boundaries of the constraint (13) and denote the two roots of the quadratic equation by b_- and b_+ ($> b_-$).

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